Linear Algebra II

06/05/2015, Monday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 $(8 + 10 = 18 \text{ pts})$ **Inner product spaces**

(a) Let V be an inner product space. Find real numbers a and b such that the so-called Apollonius' identity

$$\|z - x\|^2 + \|z - y\|^2 = a\|x - y\|^2 + b\|z - \frac{x + y}{2}\|^2$$

holds for any triple x, y, and z in V.

(b) Consider the vector space $C[-1, 1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find the best approximation of the constant function 1 within the subspace spanned by the vectors x and $|x|$.

2 $(13 + 5 = 18 \text{ pts})$ **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a singular value decomposition for M.

(b) Find the best rank 2 approximation of M.

3 $(2+4+4+4+4 = 18$ pts$)$ **Eigenvalues**

Suppose that a matrix has the characteristic polynomial

$$p(\lambda) = \lambda(\lambda+2)(\lambda^2+1).$$

Prove that this matrix is

(a) singular.

(b) diagonalizable.

(c) NOT symmetric.

(d) NOT skew-symmetric.

(e) NOT orthogonal.

4 $(8+10 = 18$ pts$)$ **Positive definiteness**

Let a be a real number. Determine all values of a such that the matrix

$$\begin{bmatrix} 1 & a & 1 \\ a & a & a+1 \\ 1 & a+1 & 1 \end{bmatrix}$$

is

(a) positive definite.

(b) negative definite.

5 $(3+5+10 = 18$ pts$)$ **Jordan canonical form**

Consider the matrix

$$\begin{bmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & -1 \\ 0 & -4 & 0 & 4 \end{bmatrix}.$$

(a) Show that the characteristic polynomial is $p(\lambda) = (\lambda-2)^{\cancel{2}\,4}$.

(b) Is it diagonalizable? Why?

(c) Put it into the Jordan canonical form.

10 pts free